

A Method for Solving Positive Pair Fully Fuzzy Matrix Equation: A Review

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Abstract

The pair matrix equation has some numerous applications in control system engineering, such as for stability analysis of linear control systems and also for reduction of nonlinear control system models. Solving pair matrix equation has intrigued many researchers due to its ability to handle imprecise information of real problems. Thus, in this work, we used the fuzzy kronecker product and vec-operator to transform the fully fuzzy pair matrix equation into a fully fuzzy linear system. Then the fully fuzzy linear system is converted to an associated linear system. The solution of the associated linear system is obtained by applying pseudoinverse method to get the final solution. Some related theoretical developments, corollary and numerical example is constructed to illustrate the algorithm.

Keywords: Fully fuzzy matrix equation, Triangular fuzzy number, Associated linear system, Kronecker product and Vec operator.

Introduction

The case of determining some vague and imprecise problems in various field of physical application has been a subject of research interest. The concept of fuzzy set and fuzzy number was introduced by Zadeh (1965) which gave an opportunity to deal with non statistical imprecise or vague matter rather than randomness. Fuzzy set is also of interest to researchers in various fields such as medicine, engineering, political science, artificial intelligence robotics, signal processing and network systems.

Much work has been focused in the study of fuzzy matrix equation (FME). (Guo and Gong 2010), used block Gaussian elimination method to solve fuzzy matrix equation $A\tilde{X}_m = \tilde{B}_m$ and the Sylvester fuzzy matrix equation $A\tilde{X} + \tilde{X}B = C$. In their work, some of the coefficients considered are fuzzy numbers. On the other hand, there are also a number of studies in which all the coefficients are in the form of fuzzy numbers. We give some examples;

- The fully fuzzy matrix equation (FFME) of $\tilde{A}_m\tilde{X}_m = \tilde{B}_m$ considered in (Otadi and Mosleh, 2012b).
- The fully fuzzy continuous-time Sylvester matrix equation of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ and $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ considered in (Daud et al., 2015, 2018b) and
- The fully fuzzy discrete time Sylvester matrix equation of $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$ considered in (Daud et al., 2018a).

Otadi and Mosleh (2012a) constructed a new method for solving an $n \times n$ fuzzy matrix equation (FME) of $A_m \tilde{X}_m = \tilde{B}_m$ whose coefficient matrix is crisp, and the right – handside matrix is an arbitrary fuzzy matrix. They used the embedding method. This method involves replacement of the original $n \times n$ fuzzy linear system by two $n \times n$ crisp linear system. It is clear that in large systems, solving linear system of $n \times n$ is better than solving $2n \times 2n$ linear system. This method restricts the coefficient of the fuzzy matrix equation to be a crisp matrix.

Also, Otadi and Mosleh (2012b) proposed a method to solve arbitrary fully fuzzy matrix equation (FFME) of the form $\tilde{A}_m \tilde{X}_m = \tilde{B}_m$, where $\tilde{A}, \tilde{B}$ are $n \times n$ arbitrary triangular fuzzy number matrices and the unknown $\tilde{X}$ is a matrix consisting of n^2 positive number. They applied linear programming techniques to obtain positive solution for arbitrary fully fuzzy matrix equation. The positivity assumption is not so restrictive in applied problems and their existing method depends on single matrix equation.

Furthermore, Malkawi et al. (2015) proposed a study to solve fully fuzzy Sylvester matrix equation which consist of the continuous time- Sylvester fully fuzzy matrix equation written in form as $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$. They used kronecker product and vec operator to transform the fully fuzzy Sylvester matrix equation to fully fuzzy linear system. This is obtained by using an equivalent linear system without any operation and classical inversion method was used to find the solution. Their study was restricted to solution of single matrix equation.

Recently, Malkawi et al. (2018a) solve fully fuzzy Sylvester matrix equation which consist of the discrete time- Sylvester fully fuzzy matrix equation written in form as $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{Q}$. They used kronecker product and vec operator and inversion method to obtain a solution. Their study also restricts to solution of single matrix equation.

Meanwhile some researcher considered the case of a pair matrix equation. Far few contribution have appeared in the literature for solving the pair matrix equation in a fuzzy environment such as in Sadeghi et al. (2011), solve the pair of fuzzy matrix equation which consist of continuous time Sylvester fuzzy matrix equation and discrete time Sylvester fuzzy matrix equation written in form as $A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1$ and $A_2\tilde{X}B_2 = \tilde{C}_2$ respectively, where the coefficients of the equations are in the form of arbitrary fuzzy numbers and crisp numbers. They used kronecker product and vec operator for transforming system of fuzzy linear equation to fuzzy linear system. A new iterative method for solving two non-square $m \times n$ system instead of one $2m \times 2n$ system was presented. But their study restricts the coefficient of the fuzzy matrix equation to crisp and fuzzy matrix.

Daud et al. (2018b) proposed a direct method to solve a positive pair fully fuzzy matrix equation (PFFME) which consist of continuous and discrete time Sylvester matrix equation written in form as $\tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1$ and $\tilde{A}_2\tilde{X}\tilde{B}_2 - \tilde{X} = \tilde{C}_2$ respectively. They used fuzzy konecker product and fuzzy vec operator to obtain a solution. Their study restricts the coefficients of the fully fuzzy matrix equation to square fully fuzzy matrix equation.

Daud et al., (2019) improved the associated linear system that was originally constructed by (Malkawi et al., 2014) to solve a pair fully fuzzy matrix equation whereby the coefficient

of the pair fully fuzzy matrix equation is square and rectangular. It consists of continuous and discrete time Sylvester matrix equation written in form as $\tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1$ and $\tilde{A}_2\tilde{X}\tilde{B}_2 = \tilde{C}_2$ respectively. They used Kaufmann approximation method, fuzzy kronecker product, fuzzy vec operator and developed a new associated linear system to obtain a solution. However, it is noted that in the aforementioned study the solution of pair fully fuzzy matrix equation where all its coefficients are fuzzy matrix and the coefficients of the fuzzy matrix equation are of different form can be solved using the same method of (Daud et al., 2018) which was stated in their paper. Thus, in this research, a review on a method for solving positive pair of fully fuzzy matrix equation (PFFME) with the examples of (Daud et al., 2019) was considered using (Daud et al., 2018) algorithm.

Preliminaries

This section will recall some definitions and basic operation of PFFME that will be used in this study.

Definition 2.1 A fuzzy number is a function such as $\mu: \mathbb{R} \rightarrow [0,1]$ satisfying the following properties:

- μ is normal, that is, there exist an $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$
- μ is fuzzy convex, that is $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$ for any $x, y \in \mathbb{R}$, $\lambda \in [0,1]$;
- μ is upper semicontinuous means that at point $x \in \mathbb{R}$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(y) < \mu(x) + \epsilon$;
- $\text{supp } \mu = \{x \in \mathbb{R} \mid \mu(x) > 0\}$ is the support of μ , and its closure $\text{cl}(\text{supp } \mu)$ is compact.

Definition 2.2 (Dubois and Prade, 1978). The arithmetic operation of two fuzzy numbers $\tilde{M} = (m, \alpha, \beta)$ and $\tilde{N} = (n, \gamma, \delta)$ are as follows:

1. Sum

$$\tilde{M} \oplus \tilde{N} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta) \quad (2)$$

2. Opposite:

$$-\tilde{M} = -(m, \alpha, \beta) = (-m, \beta, \alpha) \quad (3)$$

3. Difference

$$\tilde{M} \ominus \tilde{N} = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma) \quad (4)$$

4. Product:

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) = (mn, m\gamma + n\alpha, m\delta + n\beta) \quad (5)$$

Definition 2.3: (Dehghan et al., 2006) A positive fuzzy number $\tilde{X} = (x, y, z)$, is solution of fully fuzzy linear system, $\tilde{A}\tilde{X} = \tilde{B}$, where $\tilde{A} = (A, M, N) \geq 0$ and $\tilde{B} = (b, h, g) \geq 0$, i.e

$$(A, M, N) \otimes (x, y, z) = (b, h, g)$$

if

$$\begin{cases} Ax = b \\ Ay + Mx = h \\ Az + Nx = g \end{cases} \quad (6)$$

Definition 2.4 (Malkawi et al., 2015):- Let $[\tilde{A}] = (\tilde{a}_{ij})_{m \times n}$ be a fuzzy matrix. The transpose fuzzy matrix $\tilde{A}^T$ is denoted as follows

$$\tilde{A}^T = (\tilde{a}_{ji})_{n \times m} \quad (7)$$

Definition 2.5 (Malkawi et al., 2014): A fully fuzzy matrix equation is a matrix equation whereby the matrix and the element of the vector are fully fuzzy.

Definition 2.6 (Malkawi et al., 2015): Fuzzy kronecker product of $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{p \times q}$ is represent as

$$\begin{aligned} \tilde{A} \otimes_k \tilde{B} &= [\tilde{a}_{ij} \tilde{B}]_{(mp) \times (nq)} \\ \tilde{A} \otimes_k \tilde{B} &= \begin{bmatrix} \tilde{a}_{11} \tilde{B} & \tilde{a}_{12} \tilde{B} & \dots & \tilde{a}_{1n} \tilde{B} \\ \tilde{a}_{21} \tilde{B} & \tilde{a}_{22} \tilde{B} & \dots & \tilde{a}_{2n} \tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} \tilde{B} & \tilde{a}_{m2} \tilde{B} & \dots & \tilde{a}_{mn} \tilde{B} \end{bmatrix} \end{aligned} \quad (8)$$

Definition 2.7 (Malkawi et al., 2015): Vec-operator of a fuzzy matrix is a linear transformation that converts the fuzzy matrix $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_s)$ into a column vector

$$\text{Vec}(\tilde{C}) = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{bmatrix} \quad (9)$$

Definition 2.8: (Herdelberz, 2005) The unitary fuzzy matrix is a square matrix defined as $\tilde{U}_n = (\tilde{U}_{ij})_{n \times n}$

$$\tilde{U}_{ji} = \begin{cases} (0,0,0) & i \neq j \\ (1,0,0) & i = j \end{cases}$$

In matrix form, $\tilde{U}_n = (\tilde{U}_{ij})_{n \times n}$ is represented as follows

$$\tilde{U} = \begin{bmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{bmatrix} \quad (10)$$

Definition 2.9 (Moore et al., 1920): Pseudoinverse is a matrix inverse- like object defined for a complex matrix even if it is not necessarily square.

Theoretical development for solving the PFFME

Theorem 1: Malkawi et al. (2015). If $\tilde{A} = (A_{ij})_{m \times m}$ is a fuzzy matrix and $\tilde{U} = (\tilde{U}_{ij})_{p \times p}$ is a unitary fuzzy matrix, then

$$\begin{aligned} i. \tilde{A}\tilde{U} &= \tilde{U}\tilde{A} = \tilde{A} \\ ii. \tilde{U}^T &= \tilde{U} \end{aligned} \quad (11)$$

Proof:- i. $\tilde{A}\tilde{U} = \tilde{A}$

$$\begin{aligned} \tilde{A}\tilde{U} &= \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \dots & \tilde{a}_{nm} \end{bmatrix} \begin{bmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{a}_{11} + 0 + \dots + 0 & 0 + \tilde{a}_{12} + \dots + 0 & \dots & 0 + 0 + \dots + \tilde{a}_{1m} \\ \tilde{a}_{21} + 0 + \dots + 0 & 0 + \tilde{a}_{22} + \dots + 0 & \dots & 0 + 0 + \dots + \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} + 0 + \dots + 0 & 0 + \tilde{a}_{n2} + \dots + 0 & \dots & 0 + 0 + \dots + \tilde{a}_{nm} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \dots & \tilde{a}_{nm} \end{bmatrix} = \tilde{A} \end{aligned}$$

Hence, $\tilde{A}\tilde{U} = \tilde{A}$

Similarly, $\tilde{U}\tilde{A} = \tilde{A}$.

ii. $\tilde{U} = \tilde{U}^T$

$$\tilde{U}_{ij} = \begin{bmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{bmatrix}, \tilde{U}_{ji} = \begin{bmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{bmatrix} = \tilde{U}^T, \forall i \neq j. \text{ Q. E. D}$$

Theorem 2: Malkawi et al. (2015). Let $\tilde{P} = (\tilde{p}_{ij})_{m \times m}$, $\tilde{X} = (\tilde{x}_{ij})_{m \times n}$ and $(\tilde{U}_{ij})_{n \times n}$, where $\tilde{a}_{ij}, \tilde{x}_{ij} \in F[\mathbb{R}]$, $\tilde{U}$ is a unitary fuzzy matrix.

$$\text{Then } \tilde{P}\tilde{X} = \text{Vec}(\tilde{P}\tilde{X}) = (\tilde{U} \otimes_k \tilde{P})\text{Vec}(\tilde{X}) \quad (12)$$

Proof: Using theorem 1, $\tilde{P}\tilde{X} = \tilde{P}\tilde{X}\tilde{U}$ then $\text{Vec}(\tilde{P}\tilde{X}) = \text{Vec}(\tilde{P}\tilde{X}\tilde{U}) = (\tilde{U}_n \otimes_k \tilde{P})\text{Vec}(\tilde{X})$

However, since $\tilde{U}^T = \tilde{U}$, then $\text{Vec}(\tilde{P}\tilde{X}) = (\tilde{U}_n \otimes_k \tilde{P})\text{Vec}(\tilde{X})$ Q. E. D

Theorem 3: Malkawi et al. (2015). Let $\tilde{Q} = (\tilde{q}_{ij})_{n \times n}$, $\tilde{X} = (\tilde{x}_{ij})_{m \times n}$ and $(\tilde{U}_{ij})_{m \times m}$, where $\tilde{q}_{ij}, \tilde{x}_{ij} \in F[\mathbb{R}]$, where $\tilde{U}$ is a unitary fuzzy matrix. Then

$$\text{Vec}(\tilde{X}\tilde{Q}) = (\tilde{Q}^T \otimes_k \tilde{U}_m)\text{Vec}(\tilde{X}) \quad (13)$$

Proof: Since $\tilde{X}\tilde{Q} = \tilde{U}\tilde{X}\tilde{Q}$

$$\text{Vec}(\tilde{X}\tilde{Q}) = \text{Vec}(\tilde{U}\tilde{X}\tilde{Q}) = (\tilde{Q}^T \otimes_k \tilde{U}_m)\text{Vec}(\tilde{X}) \quad \text{Q. E. D}$$

Theorem 4: Malkawi *et al.* (2015). let $\tilde{P} = (\tilde{p}_{ij})_{m \times m}$, $\tilde{Q} = (\tilde{q}_{ij})_{n \times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{m \times n}$. Then the fully fuzzy matrix equation $\tilde{P}\tilde{X} + \tilde{X}\tilde{Q} = \tilde{C}$ can be transformed into fully fuzzy linear equation

$$[(\tilde{U}_n \otimes_k \tilde{P}) + (\tilde{Q}^T \otimes_k \tilde{U}_m)] \text{Vec}(\tilde{X}) = \text{Vec}(\tilde{C}) \quad (14)$$

where $\tilde{U}_m$ and $\tilde{U}_n$ denote the fuzzy unitary matrices order m and n , respectively.

Proof: Using Eq. (13) $\tilde{P}\tilde{X} + \tilde{X}\tilde{Q} = \tilde{C}$ can be written as $\tilde{P}\tilde{X}\tilde{U} + \tilde{U}\tilde{X}\tilde{Q} = \tilde{C}$, then

$$\text{Vec}(\tilde{P}\tilde{X}\tilde{U}) = \text{Vec}(\tilde{C})$$

$$\text{Vec}(\tilde{P}\tilde{X}\tilde{U} + \tilde{U}\tilde{X}\tilde{Q}) = \text{Vec}(\tilde{C})$$

$$\text{Vec}(\tilde{P}\tilde{X}\tilde{U}) + \text{Vec}(\tilde{U}\tilde{X}\tilde{Q}) = \text{Vec}(\tilde{C})$$

However, since $\tilde{U}^T = \tilde{U}$, then

$$(\tilde{U}^T \otimes_k \tilde{P})\text{Vec}(\tilde{X}) + (\tilde{Q}^T \otimes_k \tilde{U}_m)\text{Vec}(\tilde{X}) = \text{Vec}(\tilde{C})$$

$$[(\tilde{U}_n \otimes_k \tilde{P}) + (\tilde{Q}^T \otimes_k \tilde{U}_m)] \text{Vec}(\tilde{X}) = \text{Vec}(\tilde{C}) \quad \text{Q. E. D}$$

Theorem 5: Daud *et al.* (2019). let $\tilde{P} = (\tilde{p}_{ij})_{m \times m}$, $\tilde{Q} = (\tilde{q}_{ij})_{n \times n}$. Then

$$\tilde{P} \otimes_k \tilde{Q} = [(\tilde{P} \otimes \tilde{U})(\tilde{U} \otimes \tilde{Q})] \quad (15)$$

Theorem 6: Daud *et al.* (2019). let $\tilde{P} = (\tilde{p}_{ij})_{p \times p}$, $\tilde{Q} = (\tilde{q}_{ij})_{r \times s}$ and $\tilde{X} = (\tilde{x}_{ij})_{q \times r}$. Then the fully fuzzy matrix equation $\tilde{P}\tilde{X}\tilde{Q}$ can be transformed into fully fuzzy linear equation

$$\text{Vec}(\tilde{P}\tilde{X}\tilde{Q}) = (\tilde{P}^T \otimes_k \tilde{Q})\text{Vec}(\tilde{X}) \quad (16)$$

Proof: Use Eq. (1) to write a matrix $\tilde{X}$ as $\tilde{X} = \tilde{X}\tilde{U}$ and also from Eq. (1) is

$$\text{Vec}(\tilde{P}\tilde{X}\tilde{Q}) = \text{Vec}(\tilde{P}\tilde{X}\tilde{U}^T\tilde{Q}). \text{ By matrix transpose is } \text{Vec}(\tilde{P}\tilde{X})(\tilde{U}^T\tilde{Q}^T)$$

$$(\tilde{Q}^T\tilde{U} \otimes_k \tilde{U})\text{Vec}(\tilde{P}\tilde{X}) = (\tilde{Q}^T \otimes_k \tilde{U})\text{Vec}(\tilde{P}\tilde{X})$$

$$\text{From Eq. (2)} \quad = (\tilde{Q}^T \otimes_k \tilde{U})[(\tilde{U} \otimes_k \tilde{P})\text{Vec}(\tilde{X})]$$

$$= [(\tilde{Q}^T \otimes_k \tilde{U})(\tilde{U} \otimes_k \tilde{P})]\text{Vec}(\tilde{X})$$

$$\text{From Eq. (5)} \quad = (\tilde{Q}^T \otimes_k \tilde{P})\text{Vec}(\tilde{X})$$

Q. E. D

Therefore, the theorem is obviously proved.

Theorem 7: Malkawi *et al.* (2014) An associated linear system is a system of linear equation $AX = B$, which is associated linear system of the FFLS, $\tilde{A}\tilde{X} = \tilde{B}$, where

$$A = \begin{pmatrix} F & 0 & 0 \\ M & F & 0 \\ N & 0 & F \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} b \\ h \\ g \end{pmatrix}, \quad (17)$$

where F, M and N are rectangular matrices with common size of n , whereas x, y, z, b, h and g are vectors of $m \times n$ components.

Algorithm for solving PFFME

STEP 1: Transforming the pair fully fuzzy matrix equation PFFME to pair fully fuzzy linear

system FFLS.

Considering the corollary, "let $\tilde{A}_1 = (\tilde{a}_{1ij})_{p \times p}$, $\tilde{A}_2 = (\tilde{a}_{2ij})_{p \times p}$, $\tilde{B}_1 = (\tilde{b}_{1ij})_{q \times q}$, $\tilde{B}_2 = (\tilde{b}_{2ij})_{q \times q}$, $\tilde{X} = (\tilde{x}_{ij})_{p \times q}$. The pair fully fuzzy matrix equation (PFFME) can be expressed in the form

$$\begin{cases} \tilde{U}_{q \times q} \otimes (\tilde{A}_1 \otimes \tilde{X}) + ((\tilde{B}_1^T)_{q \times q} \otimes \tilde{U}_{p \times p}) \tilde{X} = \tilde{C}_{v1} \\ ((\tilde{B}_2^T)_{s \times q} \otimes (\tilde{A}_2)_{r \times p}) \tilde{X} = \tilde{C}_{v2} \end{cases} \quad (20)$$

where

$$\tilde{C}_{v1} = \text{Vec}(\tilde{C}_1); \quad \tilde{C}_{v2} = \text{Vec}(\tilde{C}_2); \quad \tilde{X}_v = \text{Vec}(\tilde{X})$$

Thus, let

$$\tilde{U}_{q \times q} \otimes (\tilde{A}_1)_{p \times p} + ((\tilde{B}_1^T)_{q \times q} \otimes \tilde{U}_{p \times p}) = \tilde{L}_1$$

$$((\tilde{B}_2^T)_{s \times q} \otimes (\tilde{A}_2)_{r \times p}) = \tilde{L}_2$$

then an equation of pair fully fuzzy linear system is given by

$$\tilde{L}_1 \tilde{X} = \tilde{C}_{v1}$$

$$\tilde{L}_2 \tilde{X} = \tilde{C}_{v2}$$

STEP 2: Converting the equation of pair fully fuzzy linear system in a form of associated linear system.

The conversion of pair fully fuzzy linear system in Eq. (1) is done separately due to distraction of the fuzzy operators. The addition property fails on both fuzzy coefficients $\tilde{L}_1$ and $\tilde{L}_2$. Thus, considering the fuzzy numbers, $\tilde{L}_1 = (F_1, M_1, N_1)$, $\tilde{L}_2 = (F_2, M_2, N_2)$, $\tilde{C}_{v1} = (m^{c1}, \alpha^{c1}, \beta^{c1})$, $\tilde{C}_{v2} = (m^{c2}, \alpha^{c2}, \beta^{c2})$ and $\tilde{X} = (m^x, \alpha^x, \beta^x)$, the equation can be rewritten as

$$(F_1, M_1, N_1) (m^x, \alpha^x, \beta^x) = (m^{c1}, \alpha^{c1}, \beta^{c1})$$

$$(F_2, M_2, N_2) (m^x, \alpha^x, \beta^x) = (m^{c2}, \alpha^{c2}, \beta^{c2})$$

By fuzzy arithmetic multiplication, we obtain

$$\begin{cases} F_1 m^x = m^{c1} \\ F_1 \alpha^x + M_1 m^x = \alpha^{c1} \\ F_1 \beta^x + N_1 m^x = \beta^{c1} \\ F_2 m^x = m^{c2} \\ F_2 \alpha^x + M_2 m^x = \alpha^{c2} \\ F_2 \beta^x + N_2 m^x = \beta^{c2} \end{cases} \quad (21)$$

Which can be rearranged as follow

$$\left\{ \begin{array}{l} F_1 m^x = m^{c_1} \\ F_2 m^x = m^{c_2} \\ F_1 \alpha^x + M_1 m^x = \alpha^{c_1} \\ F_2 \alpha^x + M_2 m^x = \alpha^{c_2} \\ F_1 \beta^x + N_1 m^x = \beta^{c_1} \\ F_2 \beta^x + N_2 m^x = \beta^{c_2} \end{array} \right. \quad (22)$$

Since (F_1, M_1, N_1) , $(m^{c_1}, \alpha^{c_1}, \beta^{c_1})$ and (F_2, M_2, N_2) , $(m^{c_2}, \alpha^{c_2}, \beta^{c_2})$ have different sizes respectively, it cannot be combined.

STEP 3: Composing the system in Eq.(3) to be an associated linear system of pair fully fuzzy linear system,

$$\begin{pmatrix} F_1 & 0 & 0 \\ F_2 & 0 & 0 \\ M_1 & F_1 & 0 \\ M_2 & F_2 & 0 \\ F_1 & 0 & 0 \\ F_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^{c_1} \\ m^{c_2} \\ \alpha^{c_1} \\ \alpha^{c_2} \\ \beta^{c_1} \\ \beta^{c_2} \end{pmatrix} \quad (23)$$

$$\text{Where, } F_1 = \begin{pmatrix} m_{11}^{F_1} & \dots & m_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_1} & \dots & m_{pq}^{F_1} \end{pmatrix}; \quad F_2 = \begin{pmatrix} m_{11}^{F_2} & \dots & m_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_2} & \dots & m_{pq}^{F_2} \end{pmatrix} \quad M_1 = \begin{pmatrix} \alpha_{11}^{F_1} & \dots & \alpha_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ \alpha_{p1}^{F_1} & \dots & \alpha_{pq}^{F_1} \end{pmatrix}$$

$$; M_2 = \begin{pmatrix} \alpha_{11}^{F_2} & \dots & \alpha_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \alpha_{p1}^{F_2} & \dots & \alpha_{pq}^{F_2} \end{pmatrix}, \quad N_1 = \begin{pmatrix} \beta_{11}^{F_1} & \dots & \beta_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ \beta_{p1}^{F_1} & \dots & \beta_{pq}^{F_1} \end{pmatrix}; \quad N_2 = \begin{pmatrix} \beta_{11}^{F_2} & \dots & \beta_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \beta_{p1}^{F_2} & \dots & \beta_{pq}^{F_2} \end{pmatrix}$$

$$m^{c_1} = \begin{pmatrix} m_{11}^{c_1} \\ \vdots \\ m_{pq}^{c_1} \end{pmatrix}; \quad \alpha^{c_1} = \begin{pmatrix} \alpha_{11}^{c_1} \\ \vdots \\ \alpha_{pq}^{c_1} \end{pmatrix}; \quad \beta^{c_1} = \begin{pmatrix} \beta_{11}^{c_1} \\ \vdots \\ \beta_{pq}^{c_1} \end{pmatrix};$$

$$m^{c_2} = \begin{pmatrix} m_{11}^{c_2} \\ \vdots \\ m_{pq}^{c_2} \end{pmatrix}; \quad \alpha^{c_2} = \begin{pmatrix} \alpha_{11}^{c_2} \\ \vdots \\ \alpha_{pq}^{c_2} \end{pmatrix}; \quad \beta^{c_2} = \begin{pmatrix} \beta_{11}^{c_2} \\ \vdots \\ \beta_{pq}^{c_2} \end{pmatrix}$$

$$m^x = \begin{pmatrix} m_{11}^x \\ \vdots \\ m_{pq}^x \end{pmatrix}; \quad \alpha^x = \begin{pmatrix} \alpha_{11}^x \\ \vdots \\ \alpha_{pq}^x \end{pmatrix}; \quad \beta^x = \begin{pmatrix} \beta_{11}^x \\ \vdots \\ \beta_{pq}^x \end{pmatrix}$$

Any classical linear algebra method such as inverse method, Crammers' rule, Gaussian elimination, LU decomposition or the iterative methods can be applied. In this paper, we will implement the pseudoinversion method.

Therefore,

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} F_1 & 0 & 0 \\ F_2 & 0 & 0 \\ M_1 & F_1 & 0 \\ M_2 & F_2 & 0 \\ N_1 & 0 & F_1 \\ N_2 & 0 & F_2 \end{pmatrix}^{-1} \begin{pmatrix} m^{c_1} \\ m^{c_2} \\ \alpha^{c_1} \\ \alpha^{c_2} \\ \beta^{c_1} \\ \beta^{c_2} \end{pmatrix} \quad (24)$$

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} =$$

$$\left(\begin{pmatrix} m_{11}^{F_1} & \dots & m_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_1} & \dots & m_{pq}^{F_1} \\ m_{11}^{F_2} & \dots & m_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_2} & \dots & m_{pq}^{F_2} \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} m_{11}^{c_1} \\ \vdots \\ m_{pq}^{c_1} \\ m_{11}^{c_2} \\ \vdots \\ m_{pq}^{c_2} \end{pmatrix} \\ \left(\begin{pmatrix} \alpha_{11}^{F_1} & \dots & \alpha_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ \alpha_{p1}^{F_1} & \dots & \alpha_{pq}^{F_1} \\ \alpha_{11}^{F_2} & \dots & \alpha_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \alpha_{p1}^{F_2} & \dots & \alpha_{pq}^{F_2} \end{pmatrix} \begin{pmatrix} m_{11}^{F_1} & \dots & m_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_1} & \dots & m_{pq}^{F_1} \\ m_{11}^{F_2} & \dots & m_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_2} & \dots & m_{pq}^{F_2} \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} \alpha_{11}^{c_1} \\ \vdots \\ \alpha_{pq}^{c_1} \\ \alpha_{11}^{c_2} \\ \vdots \\ \alpha_{pq}^{c_2} \end{pmatrix} \\ \left(\begin{pmatrix} \beta_{11}^{F_1} & \dots & \beta_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ \beta_{p1}^{F_1} & \dots & \beta_{pq}^{F_1} \\ \beta_{11}^{F_2} & \dots & \beta_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \beta_{p1}^{F_2} & \dots & \beta_{pq}^{F_2} \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} m_{11}^{F_1} & \dots & m_{1q}^{F_1} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_1} & \dots & m_{pq}^{F_1} \\ m_{11}^{F_2} & \dots & m_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_2} & \dots & m_{pq}^{F_2} \end{pmatrix} \right)^{-1} \begin{pmatrix} \beta_{11}^{c_1} \\ \vdots \\ \beta_{pq}^{c_1} \\ \beta_{11}^{c_2} \\ \vdots \\ \beta_{pq}^{c_2} \end{pmatrix}$$

Which is,

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m_{11}^x \\ \vdots \\ m_{pq}^x \end{pmatrix} \\ \begin{pmatrix} \alpha_{11}^x \\ \vdots \\ \alpha_{pq}^x \end{pmatrix} \\ \begin{pmatrix} \beta_{11}^x \\ \vdots \\ \beta_{pq}^x \end{pmatrix} \end{pmatrix} \quad (25)$$

Thus, the solution is given by

$$X = \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) & \cdots & (m_{1q}^x, \alpha_{1q}^x, \beta_{1q}^x) \\ \vdots & \ddots & \vdots \\ (m_{p1}^x, \alpha_{p1}^x, \beta_{p1}^x) & \cdots & (m_{pq}^x, \alpha_{pq}^x, \beta_{pq}^x) \end{pmatrix} \quad (26)$$

whereby the elements of $\tilde{X}$ matrices are positive fuzzy number.

Numerical Example

Each of the examples consist of two equations, the first equation is Continuous - time Sylvester matrix equation and second equation is Discrete-Time Sylvester matrix equation. Let

$$\begin{aligned} \tilde{P}_1 \tilde{X} + \tilde{X} \tilde{Q}_1 &= \tilde{C}_1 \\ \tilde{P}_2 \tilde{X} \tilde{Q}_2 &= \tilde{C}_2 \end{aligned}$$

be a pair of fully fuzzy matrix equation. Given the fuzzy matrix equation

$$\tilde{P}_1 = \begin{pmatrix} (12, 8, 5) & (7, 3, 3) & (11, 8, 5) \\ (9, 5, 6) & (8, 5, 7) & (9, 3, 7) \\ (10, 1, 8) & (12, 8, 6) & (8, 3, 6) \end{pmatrix}, \tilde{P}_2 = \begin{pmatrix} (9, 2, 3) & (8, 4, 3) & (9, 2, 3) \\ (8, 3, 2) & (7, 3, 8) & (11, 3, 3) \end{pmatrix}$$

$$\tilde{Q}_1 = \begin{pmatrix} (8, 3, 2) & (10, 8, 2) \\ (7, 4, 7) & (6, 4, 3) \end{pmatrix}, \tilde{Q}_2 = \begin{pmatrix} (7, 2, 2) & (10, 4, 2) & (11, 3, 5) \\ (9, 2, 1) & (7, 2, 3) & (10, 2, 9) \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} (268, 212, 173) & (296, 265, 217) \\ (290, 206, 263) & (260, 192, 206) \\ (316, 252, 262) & (318, 234, 274) \end{pmatrix},$$

$$\tilde{C}_2 = \begin{pmatrix} (2732, 2045, 2442) & (2971, 2466, 2910) & (3564, 2641, 4944) \\ (2836, 2271, 2914) & (2962, 2683, 3478) & (3686, 2914, 5685) \end{pmatrix},$$

Where $\tilde{P}_2$, $\tilde{Q}_2$, $\tilde{C}_1$ and $\tilde{C}_2$ are rectangular fully fuzzy matrices and $\tilde{P}_1$, $\tilde{Q}_1$ are fully fuzzy square matrices.

Solution:

Considering the following pair of fully fuzzy matrix equation, $\tilde{P}_1 = (\tilde{p}_1)_{p \times p}$, $\tilde{P}_2 = (\tilde{p}_2)_{r \times p}$, $\tilde{Q}_1 = (\tilde{q}_1)_{q \times q}$, $\tilde{Q}_2 = (\tilde{q}_2)_{q \times s}$ and $\tilde{C}_1 = (\tilde{c}_1)_{p \times q}$, $\tilde{C}_2 = (\tilde{c}_2)_{p \times q}$. $\tilde{U}$ is fuzzy unitary matrix.

Step 1: The pair of fully fuzzy matrix equation is converted to fully fuzzy linear system by taking the vec operator of Eq. (1) we have:

$$\text{Vec}(\tilde{P}_1 \tilde{X} + \tilde{X} \tilde{Q}_1) = \text{Vec} \tilde{C}_1$$

From proposition 4, $(\tilde{U}_2 \otimes_k \tilde{P}_1) \text{Vec}(\tilde{X}) + (\tilde{Q}_1^T \otimes_k \tilde{U}_3) \text{Vec}(\tilde{X}) = \text{Vec} \tilde{C}_1$
 $\{(\tilde{U}_2 \otimes_k \tilde{P}_1) + (\tilde{Q}_1^T \otimes_k \tilde{U}_3)\} \text{Vec}(\tilde{X}) = \text{Vec} \tilde{C}_1$

The unitary fuzzy matrix $\tilde{U}_2 = \begin{pmatrix} (1, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) \end{pmatrix}$

The kronecker product of $\tilde{U}_2 \otimes_k \tilde{P}_1$ is as follows

$$\begin{aligned} \tilde{U}_2 \otimes_k \tilde{P}_1 &= \begin{pmatrix} (1, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) \end{pmatrix} \otimes_k \begin{pmatrix} (12, 8, 5) & (7, 3, 3) & (11, 8, 5) \\ (9, 5, 6) & (8, 5, 7) & (9, 3, 7) \\ (10, 1, 8) & (12, 8, 6) & (8, 3, 6) \end{pmatrix} \\ &= \begin{pmatrix} (12, 8, 5) & (7, 3, 3) & (11, 8, 5) & (0, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (9, 5, 6) & (8, 5, 7) & (9, 3, 7) & (0, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (10, 1, 8) & (12, 8, 6) & (8, 3, 6) & (0, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (0, 0, 0) & (12, 8, 5) & (7, 3, 3) & (11, 8, 5) \\ (0, 0, 0) & (0, 0, 0) & (0, 0, 0) & (9, 5, 6) & (8, 5, 7) & (9, 3, 7) \\ (0, 0, 0) & (0, 0, 0) & (0, 0, 0) & (10, 1, 8) & (12, 8, 6) & (8, 3, 6) \end{pmatrix} \end{aligned}$$

The transpose of matrix $\tilde{Q}_1$ is, $\tilde{Q}_1^T = \begin{pmatrix} (8, 3, 2) & (10, 8, 2) \\ (7, 4, 7) & (6, 4, 3) \end{pmatrix}$

The unitary fuzzy matrix $\tilde{U}_3 = \begin{pmatrix} (1, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (1, 0, 0) \end{pmatrix}$

The kronecker product $(\tilde{Q}_1^T \otimes_k \tilde{U}_3)$ is as follows

$$(\tilde{Q}_1^T \otimes_k \tilde{U}_3) = \begin{pmatrix} (8, 3, 2) & (10, 8, 2) \\ (7, 4, 7) & (6, 4, 3) \end{pmatrix} \otimes_k \begin{pmatrix} (1, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (1, 0, 0) \end{pmatrix}$$

$$= \begin{pmatrix} (8, 3, 2) & (0, 0, 0) & (0, 0, 0) & (7, 4, 7) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (8, 3, 2) & (0, 0, 0) & (0, 0, 0) & (7, 4, 7) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (8, 3, 2) & (0, 0, 0) & (0, 0, 0) & (7, 4, 7) \\ (7, 4, 7) & (0, 0, 0) & (0, 0, 0) & (8, 3, 2) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (7, 4, 7) & (0, 0, 0) & (0, 0, 0) & (8, 3, 2) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (7, 4, 7) & (0, 0, 0) & (0, 0, 0) & (8, 3, 2) \end{pmatrix}$$

$$(U_2 \otimes_k \tilde{P}_1) + (Q_1^T \otimes_k \tilde{U}_3) = \begin{pmatrix} (20, 11, 7) & (7, 3, 3) & (11, 8, 5) & (7, 4, 7) & (0, 0, 0) & (0, 0, 0) \\ (9, 5, 6) & (16, 8, 9) & (9, 3, 7) & (0, 0, 0) & (7, 4, 7) & (0, 0, 0) \\ (10, 1, 8) & (12, 8, 6) & (16, 6, 8) & (0, 0, 0) & (0, 0, 0) & (7, 4, 7) \\ (10, 8, 2) & (0, 0, 0) & (0, 0, 0) & (18, 12, 8) & (7, 3, 3) & (11, 8, 5) \\ (0, 0, 0) & (10, 8, 2) & (0, 0, 0) & (9, 5, 6) & (14, 9, 10) & (9, 3, 7) \\ (0, 0, 0) & (0, 0, 0) & (10, 8, 2) & (10, 1, 8) & (12, 8, 6) & (14, 7, 9) \end{pmatrix}$$

Then the vec operator of fuzzy matrix $\tilde{C}_1$ is $\text{Vec}(\tilde{C}_1) = \begin{pmatrix} (268, 212, 173) \\ (290, 206, 263) \\ (316, 252, 262) \\ (296, 265, 217) \\ (260, 192, 206) \\ (318, 234, 274) \end{pmatrix}$

and the vec-operator of fuzzy matrix $\tilde{X}$ is $\text{Vec}(\tilde{X}) = \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{31}^x, \alpha_{31}^x, \beta_{31}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{32}^x, \alpha_{32}^x, \beta_{32}^x) \end{pmatrix}$

Hence, $\{(\tilde{U}_2 \otimes_k P_1) + (Q_1^T \otimes_k \tilde{U}_3) \} \text{Vec}(X) = \text{Vec} \tilde{C}_1$ is

$$\begin{pmatrix} (20, 11, 7) & (7, 3, 3) & (11, 8, 5) & (7, 4, 7) & (0, 0, 0) & (0, 0, 0) \\ (9, 5, 6) & (16, 8, 9) & (9, 3, 7) & (0, 0, 0) & (7, 4, 7) & (0, 0, 0) \\ (10, 1, 8) & (12, 8, 6) & (16, 6, 8) & (0, 0, 0) & (0, 0, 0) & (7, 4, 7) \\ (10, 8, 2) & (0, 0, 0) & (0, 0, 0) & (18, 12, 8) & (7, 3, 3) & (11, 8, 5) \\ (0, 0, 0) & (10, 8, 2) & (0, 0, 0) & (9, 5, 6) & (14, 9, 10) & (9, 3, 7) \\ (0, 0, 0) & (0, 0, 0) & (10, 8, 2) & (10, 1, 8) & (12, 8, 6) & (14, 7, 9) \end{pmatrix}$$

$$\begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{31}^x, \alpha_{31}^x, \beta_{31}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{32}^x, \alpha_{32}^x, \beta_{32}^x) \end{pmatrix} = \begin{pmatrix} (268, 212, 173) \\ (290, 206, 263) \\ (316, 252, 262) \\ (296, 265, 217) \\ (260, 192, 206) \\ (318, 234, 274) \end{pmatrix} \quad (27)$$

and also, by taking the vec operator of both sides $\tilde{P}_2 \tilde{X} \tilde{Q}_2 = \tilde{C}_2$ we have

$$\text{Vec}(\tilde{P}_2 \tilde{X} \tilde{Q}_2) = \text{Vec} \tilde{C}_2$$

from proposition 6, $(\tilde{Q}_2^T \otimes_k \tilde{P}_2) \text{Vec}(\tilde{X}) = \text{Vec} \tilde{C}_2$

$$\text{The kronecker product of } \tilde{Q}_2^T \otimes_k \tilde{P}_2 = \begin{pmatrix} (7, 2, 2) & (9, 2, 1) \\ (10, 4, 2) & (7, 2, 3) \\ (11, 3, 5) & (10, 2, 9) \end{pmatrix} \otimes_k \begin{pmatrix} (9, 2, 3) & (8, 4, 3) & (9, 2, 3) \\ (8, 3, 2) & (7, 3, 8) & (11, 3, 3) \end{pmatrix}$$

$$= \begin{pmatrix} (63, 32, 39) & (56, 44, 37) & (63, 32, 39) & (81, 36, 36) & (72, 52, 35) & (81, 36, 36) \\ (56, 37, 30) & (49, 35, 70) & (77, 43, 43) & (72, 43, 26) & (63, 41, 79) & (72, 43, 26) \\ (90, 56, 48) & (80, 72, 46) & (90, 56, 48) & (63, 32, 48) & (56, 44, 45) & (63, 32, 48) \\ (80, 62, 36) & (70, 58, 94) & (110, 74, 52) & (56, 37, 38) & (49, 35, 77) & (56, 37, 38) \\ (99, 49, 78) & (88, 68, 73) & (99, 49, 78) & (90, 38, 111) & (80, 56, 102) & (90, 38, 111) \\ (88, 57, 62) & (77, 54, 123) & (121, 66, 88) & (80, 46, 92) & (70, 44, 143) & (110, 52, 129) \end{pmatrix}$$

$$\text{The vec operator of fuzzy matrix } \tilde{C}_2 \text{ is } \text{Vec}(\tilde{C}_2) = \begin{pmatrix} (2732, 2045, 2442) \\ (2836, 2271, 2914) \\ (2872, 2466, 2910) \\ (2962, 2683, 3487) \\ (3564, 2641, 4944) \\ (3686, 2914, 5683) \end{pmatrix}$$

Therefore, $(\tilde{Q}_2^T \otimes_k \tilde{P}_2) \text{Vec}(X) = \text{Vec}(\tilde{C}_2)$ is

$$\begin{pmatrix} (63, 32, 39) & (56, 44, 37) & (63, 32, 39) & (81, 36, 36) & (72, 52, 35) & (81, 36, 36) \\ (56, 37, 30) & (49, 35, 70) & (77, 43, 43) & (72, 43, 26) & (63, 41, 79) & (72, 43, 26) \\ (90, 56, 48) & (80, 72, 46) & (90, 56, 48) & (63, 32, 48) & (56, 44, 45) & (63, 32, 48) \\ (80, 62, 36) & (70, 58, 94) & (110, 74, 52) & (56, 37, 38) & (49, 35, 77) & (56, 37, 38) \\ (99, 49, 78) & (88, 68, 73) & (99, 49, 78) & (90, 38, 111) & (80, 56, 102) & (90, 38, 111) \\ (88, 57, 62) & (77, 54, 123) & (121, 66, 88) & (80, 46, 92) & (70, 44, 143) & (110, 52, 129) \end{pmatrix} \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{31}^x, \alpha_{31}^x, \beta_{31}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{32}^x, \alpha_{32}^x, \beta_{32}^x) \end{pmatrix} \\ = \begin{pmatrix} (2732, 2045, 2442) \\ (2836, 2271, 2914) \\ (2872, 2466, 2910) \\ (2962, 2683, 3487) \\ (3564, 2641, 4944) \\ (3686, 2914, 5683) \end{pmatrix} \quad (28)$$

Hence the PFFME is transferred to the following PFFLS.

Step 2: We now convert the pair fully fuzzy linear system to associated linear system.

In order to achieve an associated linear system, all matrices obtained in Eq. (27) and Eq. (28) need to be converted to crisp form of matrices. Then, Eq. (27) yields:

$$F_1 = \begin{pmatrix} 20 & 7 & 11 & 7 & 0 & 0 \\ 9 & 16 & 9 & 0 & 7 & 0 \\ 10 & 12 & 16 & 0 & 0 & 7 \\ 10 & 0 & 0 & 18 & 7 & 11 \\ 0 & 10 & 0 & 9 & 14 & 9 \\ 0 & 0 & 10 & 10 & 12 & 14 \end{pmatrix}, M_1 = \begin{pmatrix} 11 & 3 & 8 & 4 & 0 & 0 \\ 5 & 8 & 3 & 0 & 4 & 0 \\ 1 & 8 & 6 & 0 & 0 & 4 \\ 8 & 0 & 0 & 12 & 3 & 8 \\ 0 & 8 & 0 & 5 & 9 & 3 \\ 0 & 0 & 8 & 1 & 8 & 7 \end{pmatrix}, N_1 = \begin{pmatrix} 7 & 3 & 5 & 7 & 0 & 0 \\ 6 & 9 & 7 & 0 & 7 & 0 \\ 8 & 6 & 8 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 & 3 & 5 \\ 0 & 2 & 0 & 6 & 10 & 7 \\ 0 & 0 & 2 & 8 & 6 & 9 \end{pmatrix}$$

and $\text{Vec}(\tilde{C}_1)$ yields

$$m^{c_1} = \begin{pmatrix} 268 \\ 290 \\ 316 \\ 296 \\ 260 \\ 318 \end{pmatrix}, \quad \alpha^{c_1} = \begin{pmatrix} 212 \\ 206 \\ 252 \\ 265 \\ 192 \\ 234 \end{pmatrix}, \quad \beta^{c_1} = \begin{pmatrix} 173 \\ 263 \\ 262 \\ 217 \\ 206 \\ 274 \end{pmatrix}$$

On the other hand, the crisp matrices extracted from Eq. (28) yields

$$F_2 = \begin{pmatrix} 63 & 56 & 63 & 81 & 72 & 81 \\ 56 & 49 & 77 & 72 & 63 & 63 \\ 90 & 80 & 90 & 63 & 56 & 63 \\ 80 & 70 & 110 & 56 & 49 & 77 \\ 99 & 88 & 99 & 90 & 80 & 90 \\ 88 & 77 & 121 & 80 & 70 & 110 \end{pmatrix}, M_2 = \begin{pmatrix} 32 & 44 & 32 & 36 & 56 & 36 \\ 37 & 35 & 43 & 43 & 41 & 49 \\ 56 & 72 & 56 & 32 & 44 & 32 \\ 62 & 58 & 74 & 37 & 35 & 43 \\ 49 & 68 & 49 & 38 & 56 & 38 \\ 57 & 54 & 66 & 46 & 44 & 52 \end{pmatrix}, N_2 = \begin{pmatrix} 39 & 37 & 39 & 36 & 35 & 36 \\ 30 & 70 & 43 & 26 & 79 & 38 \\ 48 & 46 & 48 & 48 & 45 & 48 \\ 36 & 94 & 52 & 38 & 77 & 54 \\ 78 & 73 & 78 & 111 & 102 & 111 \\ 62 & 123 & 88 & 92 & 143 & 129 \end{pmatrix}$$

and $\text{Vec}(\tilde{C}_2)$ yields

$$m^{c_2} = \begin{pmatrix} 2732 \\ 2836 \\ 2872 \\ 2962 \\ 3564 \\ 3686 \end{pmatrix}, \quad \alpha^{c_2} = \begin{pmatrix} 2045 \\ 2271 \\ 2466 \\ 2683 \\ 2641 \\ 2914 \end{pmatrix}, \quad \beta^{c_2} = \begin{pmatrix} 2442 \\ 2914 \\ 2910 \\ 3846 \\ 4944 \\ 5685 \end{pmatrix}$$

Step 3: Thus, the crisp matrix is composed to an associated linear system,

$$SX = B, \begin{pmatrix} F_i & 0 & 0 \\ M_i & F_i & 0 \\ N_i & 0 & F_i \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^{c_i} \\ \alpha^{c_i} \\ \beta^{c_i} \end{pmatrix},$$

Composing crisp matrix to an associated linear system (for example 1).

$\left(\begin{array}{cccccc cccccc cccccc}20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\9 & 16 & 11 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\63 & 56 & 53 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\\hline11 & 3 & 8 & 4 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\5 & 8 & 3 & 0 & 4 & 0 & 9 & 16 & 11 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\1 & 8 & 6 & 0 & 0 & 4 & 10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\8 & 0 & 0 & 12 & 3 & 8 & 10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 \\0 & 8 & 0 & 5 & 9 & 3 & 0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 \\0 & 0 & 8 & 1 & 8 & 7 & 0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 \\32 & 44 & 32 & 36 & 52 & 36 & 63 & 56 & 53 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 \\37 & 35 & 43 & 43 & 41 & 49 & 56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 \\56 & 72 & 56 & 32 & 44 & 32 & 90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 \\62 & 58 & 74 & 37 & 35 & 43 & 80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 \\49 & 68 & 49 & 38 & 56 & 38 & 99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 \\57 & 54 & 66 & 46 & 44 & 52 & 88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 \\\hline7 & 3 & 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 \\6 & 9 & 7 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 16 & 11 & 0 & 7 & 0 \\8 & 6 & 8 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 12 & 16 & 0 & 0 & 7 \\2 & 0 & 0 & 8 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 18 & 7 & 11 \\0 & 2 & 0 & 6 & 10 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 9 & 14 & 9 \\0 & 0 & 2 & 8 & 6 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 12 & 14 \\39 & 37 & 39 & 36 & 35 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 63 & 56 & 53 & 81 & 72 & 81 \\30 & 70 & 33 & 26 & 79 & 38 & 0 & 0 & 0 & 0 & 0 & 0 & 56 & 49 & 77 & 72 & 63 & 99 \\48 & 46 & 48 & 48 & 45 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 90 & 80 & 90 & 63 & 56 & 63 \\36 & 94 & 52 & 38 & 77 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 70 & 110 & 56 & 49 & 77 \\78 & 73 & 78 & 111 & 102 & 111 & 0 & 0 & 0 & 0 & 0 & 0 & 99 & 88 & 99 & 90 & 80 & 90 \\62 & 123 & 88 & 92 & 143 & 129 & 0 & 0 & 0 & 0 & 0 & 0 & 88 & 77 & 121 & 80 & 70 & 110 \end{array}\right)$	$\left(\begin{array}{c}m_{11}^x \\m_{21}^x \\m_{12}^x \\m_{22}^x \\m_{13}^x \\m_{23}^x \\\alpha_{11}^x \\\alpha_{21}^x \\\alpha_{12}^x \\\alpha_{22}^x \\\alpha_{13}^x \\\alpha_{23}^x \\\beta_{11}^x \\\beta_{21}^x \\\beta_{12}^x \\\beta_{22}^x \\\beta_{13}^x \\\beta_{23}^x\end{array}\right)$	$=$	$\left(\begin{array}{c}268 \\290 \\316 \\296 \\260 \\318 \\2732 \\2836 \\2872 \\2962 \\3564 \\3686 \\212 \\206 \\252 \\265 \\192 \\234 \\2045 \\2271 \\2466 \\2683 \\2641 \\2914 \\173 \\263 \\262 \\217 \\206 \\274 \\2442 \\2914 \\2910 \\3846 \\4944 \\5685\end{array}\right)$
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Using proposition 7, Singular Value Decomposition SDV of $F \neq 0$ then SDV of $S \neq 0$. The SVD is used to get the scalar value of a rectangular matrix. Hence the solution is obtained by $X = S^{-1}B$.

Inversion of the associated linear system (for example 1)

$$\begin{pmatrix} m_{11}^x \\ m_{21}^x \\ m_{31}^x \\ m_{12}^x \\ m_{22}^x \\ m_{13}^x \\ m_{23}^x \\ \alpha_{11}^x \\ \alpha_{21}^x \\ \alpha_{12}^x \\ \alpha_{22}^x \\ \alpha_{13}^x \\ \alpha_{23}^x \\ \beta_{11}^x \\ \beta_{21}^x \\ \beta_{12}^x \\ \beta_{22}^x \\ \beta_{13}^x \\ \beta_{23}^x \end{pmatrix} = \begin{pmatrix} 20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 16 & 11 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 63 & 56 & 53 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 11 & 3 & 8 & 4 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 3 & 0 & 4 & 0 & 9 & 16 & 11 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 6 & 0 & 0 & 4 & 10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 12 & 3 & 8 & 10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 5 & 9 & 3 & 0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 1 & 8 & 7 & 0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 44 & 32 & 36 & 52 & 36 & 63 & 56 & 53 & 81 & 72 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 37 & 35 & 43 & 43 & 41 & 49 & 56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 56 & 72 & 56 & 32 & 44 & 32 & 90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 62 & 58 & 74 & 37 & 35 & 43 & 80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 49 & 68 & 49 & 38 & 56 & 38 & 99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 57 & 54 & 66 & 46 & 44 & 52 & 88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 7 & 3 & 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 7 & 11 & 7 & 0 & 0 & 0 & 0 \\ 6 & 9 & 7 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 16 & 11 & 0 & 7 & 0 & 0 & 0 \\ 8 & 6 & 8 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 12 & 16 & 0 & 0 & 7 & 0 & 0 \\ 2 & 0 & 0 & 8 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 18 & 7 & 11 & 0 & 0 \\ 0 & 2 & 0 & 6 & 10 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 9 & 14 & 9 & 0 & 0 \\ 0 & 0 & 2 & 8 & 6 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 12 & 14 & 0 & 0 \\ 39 & 37 & 39 & 36 & 35 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 63 & 56 & 53 & 81 & 72 & 81 & 0 & 0 \\ 30 & 70 & 33 & 26 & 79 & 38 & 0 & 0 & 0 & 0 & 0 & 0 & 56 & 49 & 77 & 72 & 63 & 99 & 0 & 0 \\ 48 & 46 & 48 & 48 & 45 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 90 & 80 & 90 & 63 & 56 & 63 & 0 & 0 \\ 36 & 94 & 52 & 38 & 77 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 70 & 110 & 56 & 49 & 77 & 0 & 0 \\ 78 & 73 & 78 & 111 & 102 & 111 & 0 & 0 & 0 & 0 & 0 & 0 & 99 & 88 & 99 & 90 & 80 & 90 & 0 & 0 \\ 62 & 123 & 88 & 92 & 143 & 129 & 0 & 0 & 0 & 0 & 0 & 0 & 88 & 77 & 121 & 80 & 70 & 110 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 268 \\ 290 \\ 316 \\ 296 \\ 260 \\ 318 \\ 2732 \\ 2836 \\ 2872 \\ 2962 \\ 3564 \\ 3686 \\ 212 \\ 206 \\ 252 \\ 265 \\ 192 \\ 234 \\ 2045 \\ 2271 \\ 2466 \\ 2683 \\ 2641 \\ 2914 \\ 173 \\ 263 \\ 262 \\ 217 \\ 206 \\ 274 \\ 2442 \\ 2914 \\ 2910 \\ 3846 \\ 4944 \\ 5685 \end{pmatrix}$$

Step 4: Solution obtained using pseudoinverse method (from example 1)

$$\begin{pmatrix} m_{11}^x \\ m_{21}^x \\ m_{31}^x \\ m_{12}^x \\ m_{22}^x \\ m_{32}^x \\ \alpha_{11}^x \\ \alpha_{21}^x \\ \alpha_{12}^x \\ \alpha_{22}^x \\ \alpha_{13}^x \\ \alpha_{23}^x \\ \beta_{11}^x \\ \beta_{21}^x \\ \beta_{12}^x \\ \beta_{22}^x \\ \beta_{13}^x \\ \beta_{23}^x \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 7 \\ 6 \\ 4 \\ 10 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 6 \\ 2 \end{pmatrix}$$

Thus, the $\text{vec}(\tilde{X})$ is

$$\text{vec}(\tilde{X}) = \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) \\ (m_{31}^x, \alpha_{31}^x, \beta_{31}^x) \\ (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{32}^x, \alpha_{32}^x, \beta_{32}^x) \end{pmatrix} = \begin{pmatrix} (5, 1, 1) \\ (7, 1, 0) \\ (7, 2, 1) \\ (6, 1, 2) \\ (4, 0, 1) \\ (10, 3, 4) \end{pmatrix}$$

Then, the fuzzy matrix solution is,

$$\tilde{X} = \begin{pmatrix} (m_{11}^x, \alpha_{11}^x, \beta_{11}^x) & (m_{12}^x, \alpha_{12}^x, \beta_{12}^x) \\ (m_{21}^x, \alpha_{21}^x, \beta_{21}^x) & (m_{22}^x, \alpha_{22}^x, \beta_{22}^x) \\ (m_{31}^x, \alpha_{31}^x, \beta_{31}^x) & (m_{32}^x, \alpha_{32}^x, \beta_{32}^x) \end{pmatrix}$$

$$\tilde{X} = \begin{pmatrix} (5, 1, 1) & (6, 1, 2) \\ (7, 1, 0) & (4, 0, 1) \\ (7, 2, 1) & (10, 3, 4) \end{pmatrix}$$

Conclusion

Many researchers have conducted studies on methods for solving pair fully fuzzy matrix equation when the fuzzy coefficients are of the same size but few studies have attempt to construct method by associated linear systems when the fuzzy coefficients are of the different size, whereby most method employed linear programming techniques and embedding approaches for solving pair fully fuzzy matrix equations.

Thus, this study attempts to improve on (Daud et al., 2019) by using (Daud et al., 2018) algorithm to solve the numerical application. Fuzzy kronecker product, fuzzy vec- operator and associated linear system is used. Then the final solution was obtained using pseudoinverse method.

Recommendations

We now turn our attention to comply with future problems that can be done.

1. We solved the pair fully fuzzy matrix equation when the matrix is non singular value decomposition. Therefore, it is suggested to solve pair fully fuzzy matrix equation when the matrix is singular value decomposition.
2. We followed (m, α, β) form for triangular fuzzy numbers, it is suggested to formulate methods that can deal with other forms like (a, b, c) .
3. We implemented pseudoinverse to get the solution, it is suggested to implement any other suitable method.

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